1041.**Proposed by George Apostolopoulos**, **Messolonghi**, **Greece**.

Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$(1-2a)^5+(1-2b)^5+(1-2c)^5+\frac{80}{3}abc \leq 1.$$

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.

Let $S_n := (1-2a)^n+(1-2b)^n+(1-2c)^n, n \in \mathbb{N} \cup \{0\}$ and $p := ab+bc+ca, q := abc$.

$S_0 = 3, S_1 = (1-2a)+(1-2b)+(1-2c) = 1, S_2 = (1-2a)^2+(1-2b)^2+(1-2c)^2 =$
$4(a^2+b^2+c^2)-4(a+b+c)+3 = 4(1-2p)-1 = 3-8p$.

Since $(1-2a)(1-2b)+(1-2b)(1-2c)+(1-2c)(1-2a) = 4(ab+ac+bc)-$
$4(a+b+c)+3 =$
$4p-1, (1-2a)(1-2b)(1-2c) = 4(ab+ac+bc)-2(a+b+c)-8abc+1 = 4p-8q-1$
then for calculation of S_n we have the following recurrence

(**1**) $\quad S_{n+1} = S_n-(4p-1)S_{n-1}+(4p-8q-1)S_{n-2}, n \geq 2$

with initial condition $S_0 = 3, S_1 = 1, S_2 = 3-8p$.

Using recurrence (**1**) we consecutively obtain

$S_3 = (3-8p)-(4p-1)\cdot 1+(4p-8q-1)\cdot 3 = 1-24q$,

$S_4 = 1-24q-(4p-1)(3-8p)+(4p-8q-1)\cdot 1 = 32p^2-16p-32q+3$,

$S_5 = 32p^2-16p-32q+3-(4p-1)(1-24q)+(4p-8q-1)(3-8p) =$
$160pq-80q+1$ and original unequally becomes

$$160pq-80q+1+\frac{80q}{3} \leq 1 \iff 0 \leq q(1-3p),$$

where latter inequality holds because $3p = 3(ab+bc+ca) \leq (a+b+c)^2 = 1$.